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Uncertainty Calculations in Flow Measurement

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Chapter 1 Introduction

When a user buys an expensive flow metering system, the single most important objective is to measure the required flow value(s) with demonstrable accuracy – and to keep doing do so. The uncertainty calculations are therefore as much a key part of a metering system as its physical components. The principles of uncertainty calculations are not hard to follow for those who are comfortable with calculus and statistics, but the practicalities justify their reputation for being difficult. As The GUM (3.4.8), which we will meet in a moment, puts it:

The evaluation of uncertainty is neither a routine task nor a purely mathematical one; it depends on detailed knowledge of the nature of the [quantity measured] and of the measurement. The quality and utility of the uncertainty quoted for the result of a measurement therefore ultimately depend on the understanding, critical analysis, and integrity of those who contribute to the assignment of its value.

Although uncertainty is evidently not a purely mathematical business, it certainly involves a lot of maths. Some engineers are rather rusty in maths, and even engineers who are adept at it may have limited experience with advanced statistics. The best that we can do here is to introduce the relevant ideas as informally and intuitively as possible, and postpone detailed mathematical formulation for as long as possible. In any case, adopting an intuitive approach may promote the understanding and critical analysis that The GUM prefers.

1.1 Measurements

Everyday life is full of measurements: the ruler in the desk drawer, the scales and measuring jug in the kitchen, the electricity meter in its cupboard and the petrol pump at the filling station to mention just a few. We all understand that none of these measurements is exact, but we are happy enough with the kitchen scales if the sponge cake rises properly.

When it comes to science and technology, measurement requires more care. Suppose a new scientific experiment makes some measurement that seems to contradict the currently understood principles of physics. Do physicists need to investigate what is going on, and perhaps come up with something like relativity or quantum mechanics as a consequence? They will need some convincing, so the first question will be: how sure can we be about that measurement?

One thing that is certain is that every measured value is subject to some error. However, if it were known that a measurement gave a positive error of 0.05% then this amount could be subtracted from the result to leave no error at all. In other words, the actual error has to be unknown or it could be eliminated. The best that the experimenter can do is therefore to try to make some helpful comments about what the actual error <u>might</u> be. As Introduction to The GUM (JCGM 104 3.8) points out, it is not even possible to say absolutely how well a value is known: the best that anybody can do is to say how well they believe it is known.

For example, at the time of writing the NIST web site quotes the gas constant, R, rather like this:

$$R = 8.314\,4621 \pm 0.000\,0075 \text{ J mol}^{-1}\text{ K}^{-1}$$

The first value represents the best estimate, and the second says that NIST is pretty sure that the true value is within the range given. “Pretty sure?” Well, yes – the whole point is that the actual error is unknown, and all that anybody can say is that they believe it is more likely to be in one range and less likely to be in another. NIST is about 68% certain that the true value lies within the given range of the best estimate, slightly better odds than flipping a coin, because the figure they have quoted is actually the Standard Uncertainty. For reasons that will become clear, this implies that there is about a 95% chance that the true value falls within twice this range, and a 99.9936% chance that it falls within four times this range.

To put it another way, according to NIST’s assessment there is about a 0.006% chance that the true gas constant is more than 0.000 03 J mol^{-1} K^{-1}different from the commonly accepted value. By comparison, the chances of winning the jackpot in the UK national lottery in a given week are about 0.000 007%. Winning the lottery is not likely but it does happen to somebody most weeks, so it is unlikely but not impossible that the accepted value of R really is that far out. In any case, please bear in mind that the quoted uncertainty itself is only a best estimate: as we shall see, uncertainty itself is uncertain.

To sum up, the “true value” is never exactly known – and because of that, the “error” can never be known either. It may help to get into the habit of mentally prefixing both these terms with “estimated”, or even “unknown”, wherever they are encountered.

1.2 Assessing Measurements

How is it possible to estimate the uncertainty in the result of a measurement? Clearly the nature of the measurement and the equipment used in it will give some useful clues. When measuring the voltage in a circuit where no great precision is required, it will not be necessary to look beyond the specification of the voltmeter.

More care is needed when measuring the bore of an orifice plate to a precision of a few micrometres. The measuring device used will clearly have an uncertainty, usually specified by the manufacturer and hopefully maintained by regular calibration at a suitable laboratory. In addition, the relevant dimensions will change with temperature and it may be necessary to correct the result for thermal expansion of both the plate and the device used to measure it. Neither the exact temperature nor its precise effect will be known, so it will be necessary to evaluate the additional uncertainty from this source and to work out how to incorporate it into the overall uncertainty of the measurement.

Suppose that the measured orifice plate is now installed in a metering run and used to determine the mass flow rate. The situation becomes much more complex, because the calculated result will depend on the orifice bore, the differential pressure generated, the density of the fluid, and a good many more things besides – each of which is measured with its own uncertainty. Combining all of these uncertainties requires an effort, but that is the routine part. Identifying all the individual sources of uncertainty and tracking down credible values is the bit that requires understanding, critical analysis and integrity – not to mention a certain amount of ingenuity from time to time.

Another useful source of information about the uncertainty of a measurement is to repeat it several times, if the nature of the measurement permits. The orifice bore is a static measurement that it would be possible to repeat time and again. On the other hand there is usually no way to know or control the precise flow through an orifice metering run, so repeat measurements are impractical.

Repeat measurements are interesting because any measurement that is sufficiently sensitive will give a slightly different result every time it is made. The scatter of the results does not tell the whole story, because there could be a fixed offset in every measurement. However, if repeated results give a wide scatter then the measurement is clearly more suspect than if they are close together. Excessive scatter may also point to a flaw in the measurement that needs to be investigated (for example, a leaking valve when proving a meter). If sufficient repeats were taken, any symmetrical scatter would eventually average out to zero – but the number of repeats is usually limited by practical and/or economic considerations.

In the past, “locked in” errors that were unknown but fixed were described as Systematic, and uncertainties resulting from scattered repeat measurements were known as Random. This terminology is still sometimes used, but widely considered outdated. That is because the orifice bore measurement that we considered would have included a **systematic component** arising, for example, from the latest calibration of the measuring device and a **random component** arising from scattered repeats. However, the best estimate of the orifice bore given on the resulting calibration certificate is “locked in” and from then on the figure has only a systematic error. The same value therefore contains a random component at one point, but not at another. For this reason, the more recent uncertainty standards talk about the way that an uncertainty is assessed at various points along the measurement chain, rather than using language that implies that a value is intrinsically subject to one sort of uncertainty or another.

1.3 Bias

We will consider here sources of measurement error that are believed to have an average value of zero. In all but a few unusual cases, an error near zero will be at least as likely as any other. Multiple separate errors of this sort will most likely partly cancel each other out, and errors in individual measurements will tend to even out over time.

When an effect that causes a steady measurement offset is either missed or intentionally left uncorrected, it leads to measurement bias. Bias does not cancel out over time, and the statistical methods that apply to uncertainty cannot be used to handle it. More worryingly in a flow measurement system that provides figures used in large financial transactions, measurement bias will steadily accumulate over long periods. Even a small bias can therefore cost somebody a significant amount of money in the end.

Measurement offsets do not have to be steady to create bias. An offset that is frequently positive and only occasionally negative has a similar net effect. Even a symmetrical offset in one contribution to a measurement can still cause bias if it has a significantly non-linear effect on the result. This applies, for example, to pulsating flow through orifice plates.

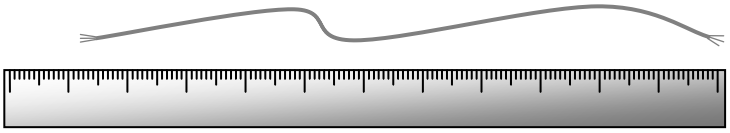
Uncertainty is inevitable in all forms of measurement, and it is usual to specify an acceptable level for it. However, the provision of an uncertainty allowance should not lead anyone to believe that bias is acceptable. A well-considered measurement specification will explicitly state that all identifiable forms of bias must be sought out and eliminated.

1.4 A Note about the Exercises

The exercises are not an optional extra! Some of them lead to conclusions whose importance will become clear in later chapters. Others build up a toolkit that will be useful when you perform uncertainty calculations yourself, or review the calculations of others.

1.5 Exercise

Exercise 1.1 – How long is a Piece of String[1] (Part 1)



A piece of string about 250mm long is measured using a metal ruler with 1mm increments. Working as a group:

1) Brainstorm the potential sources of measurement uncertainty.

2) Group the sources of uncertainty according to whether you would expect them to:

a) Be the same for every measurement over a reasonably extended period

b) Change between measurements made in quick succession

c) Change between measurements made on different days

[1] With acknowledgement for the general idea to Martin Basil, SOLV Ltd, from a paper presented to the Production and Upstream Flow measurement Workshop, 2008.

Chapter 2 Uncertainty Standards

Combining the various uncertainties in the input quantities to a measurement such as an orifice run in the proper way to determine the output uncertainty is a complex business, so it will be very useful to have some authoritative advice on how to do it properly. What is more, as large sums of money frequently change hands based on the reading from flow metering runs, interested parties will normally insist that the uncertainty analysis is performed to a recognised and respected standard.

This section provides a brief introduction to the main standards that will be explored in more or less detail in the rest of this course.

2.1 JCGM 100: The GUM

The main reference for this course is a standard published by the Joint Committee for Guides in Metrology, JCGM 100 (2008) "Evaluation of measurement data — Guide to the expression of uncertainty in measurement", generally abbreviated to The GUM. Related JCGM standards are described as Supplements or something similar. Throughout this document, "The GUM" means JCGM 100.

The GUM has the enormous advantage of being freely accessible on the web, using the following link at the time of writing:

http://www.bipm.org/utils/common/documents/jcgm/JCGM_100_2008_E.pdf

The GUM defines an analytical approach to combining uncertainties. The principle of the analytical approach is that the uncertainties of the inputs to a measurement process are assessed, and the uncertainty in the resulting figure is calculated using methods derived from mathematical statistics.

The JCGM aims The GUM at the widest possible user base, and states among its objectives that its target audience includes QC and QA people, scientific researchers, calibration laboratories and the custodians of national standards. Niche activities such as flow measurement are well below their radar. However, the principles of uncertainty analysis are the same regardless of the application.

All of the following organisations are members of the JCGM, contribute to its standards, and most importantly (should) follow The GUM in creating standards of their own:

- BIPM: Bureau International des Poids et Mesures
- OIML: Organisation Internationale de Métrologie Légale
- ISO: International Organization for Standardization
- IEC: International Electrotechnical Commission
- IUPAC: International Union of Pure and Applied Chemistry
- IUPAP: International Union of Pure and Applied Physics
- IFCC: International Federation of Clinical Chemistry and Laboratory Medicine
- ILAC: International Laboratory Accreditation Cooperation

Chapter 3 First Steps in Uncertainty

Following standards is all very well but a reasonable level of understanding of the principles behind them is always useful, and in view of the extract from JCGM 100 quoted at the start of Chapter 1 almost obligatory where uncertainty calculations are involved. This chapter therefore introduces some key concepts using a constructed example, based on fundamental statistical principles. There are few references to uncertainty standards: that comes later. The objective is give a “feel” for how uncertainty works, and to demonstrate that there is nothing magical about it.

The less obvious uncertainty distributions in this chapter have been constructed using large numbers of random numbers generated in Excel (in effect, a Monte Carlo method). However, because of the simple nature of the error distribution chosen, they could alternatively have been determined analytically (see Chapter 8).

This chapter is actually as much an exercise in designing a measurement as it is in uncertainty, which reflects the way that things should ideally be done. A good supplier will not just use uncertainty calculations to rubber stamp a completed design. They should preferably form part of the design process as a means of identifying potential problems, refining the design and so constructing the best possible system for a particular application. Such a process need not necessarily add to the cost of the system: an early uncertainty analysis can help to identify overspecified equipment as well as underspecified, and may also pick up design problems before they become expensive to rectify. However, it has to be admitted that uncertainty estimates are themselves rather uncertain and even more so in the early stages of a project where limited information is available.

3.1 A Calibration Problem

Suppose that it is necessary to measure the current in an instrument loop to within ±0.05% in order to calibrate an instrument, and it has to be done right now using whatever equipment is at hand. A digital multimeter is available, but its manual specifies that the “accuracy” of the current range as ±0.1%. The current range is not good enough, which used to be quite a common problem though it is less so now. The multimeter uses an extremely low value internal resistor to convert current to voltage, which is difficult make to a tight specification.

Several resistors marked 100Ω ±0.05% are available, and the multimeter manual specifies that its voltage range has an “accuracy” of ±0.01%. It is therefore possible to get closer to the required measurement performance by inserting a resistor in the loop and measuring the voltage across it.

We are unlikely to know details of how the resistors are manufactured in real life, but for the purposes of this example we will assume that we do. This particular manufacturer makes batches of resistors, and checks them against an instrument that measures resistance with an accuracy that is near enough to perfect for practical purposes. It selects resistors within ±0.05% of the target value for its highest grade, and classifies the rest into ±0.1% and ±0.2% grades. If a resistor measures 100.0501Ω it is assigned to a lower grade, but 100.0499Ω goes into the top grade. This is an example of a “pass/fail” calibration, which is quite common in practice.

Chapter 4 Combining Uncertainties

This section builds up to the detailed requirements and procedures of The GUM for combining uncertainty estimates via some background detail. For the time being, we will assume that necessary data such as equations and input uncertainties are known. Dealing with practical situations where this information has to be gathered is a separate adventure.

4.1 Type A and Type B Assessment

We have already seen that an uncertainty component that starts as a random measurement scatter will sometimes become locked in at some point in the measurement chain and so become "systematic". The GUM therefore classifies uncertainty assessments rather than assigning types to uncertainties themselves.

The two types of uncertainty assessment defined by The GUM are:

- **Type A assessment**, which covers the effects of a limited number of scattered repeat measurements on the uncertainty of the resulting best estimate
- **Type B assessment**, which covers assembling input error functions in order to estimate the uncertainty of the resulting output

Type B assessment is the most common in flow measurement uncertainty calculations, and we will therefore consider it first.

The effects of repeatability, requiring Type A assessment, will often have been taken into account during instrument calibration. Type A evaluation will also be relevant where calibrations with repeat runs are performed during system operation, the most obvious example being meter proving. In flow measurement it generally leads to a smaller contribution than Type B, so we consider it towards the end of the chapter.

4.2 Basic Concepts

A common requirement in flow measurement is to multiply volume flow in m^3/h by density in kg/m^3 to calculate mass flow in t/h:

$$q_m = \frac{\rho \cdot q_v}{1000}$$

For example, we will assume that density and volume flow are measured using separate instruments, each with its own error function. An assessment of the process conditions and the instruments has concluded that the results of measuring the two input parameters are normally distributed with the following parameters:

Chapter 5 Input Uncertainties

The main input uncertainties to a flow calculation will generally need to be assembled from several subcomponents. We now have all the necessary tools to do this, and there are enough detailed considerations to fill a chapter. The required output quantity from this type of process is clear, and the sensitivity coefficients are generally one. The more complex final assembly, which includes evaluation conditions, is covered in the next chapter. For now, we will assume evaluation conditions as we go – including an ambient temperature of 30ºC.

5.1 General

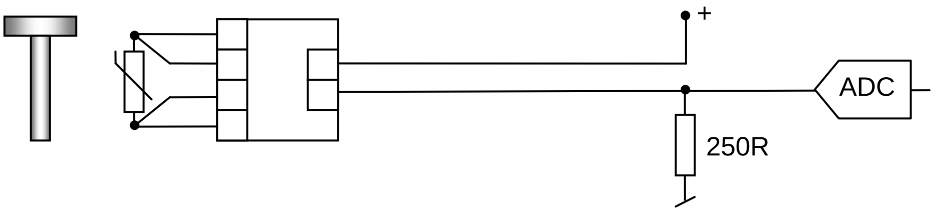
A key point to bear in mind from the outset is that sooner or later, somebody will need to understand what you have done. A knowledgeable reviewer may need to be satisfied that the calculations have been performed properly. An end user may need to update the uncertainty calculations to take account of changes to the system. Perhaps the best reason for clearly documenting what you have done is that you may return to your own calculations one day.

It is therefore a good idea to explain where the values used have come from. For example, if instrument specification documents have been used they should at least be referenced – complete with a revision code or date. The model code of each instrument should be quoted, at least as far as it affects uncertainty, along with its calibrated range.

It will be helpful for your own clarity of thought to print off relevant pages from documents, highlight the relevant figures, and keep them in a file along with the resulting calculations. These pages could then be scanned, and appended to the calculations. If copies of specifications are available as unlocked PDF files, it is even more convenient to create highlighted and searchable documents electronically using Acrobat. Bear in mind, though, that there may be copyright issues.

5.2 Temperature

Temperature brings out several points that are general to all process signals, and will be examined in some detail. We will assume for the moment that it is possible to insert a thermowell at exactly the point of interest, although it is often not so in practice. The complete loop is illustrated below.



The outside of a thermowell is at a temperature close to the required process value. An element inside the thermowell converts the temperature to resistance, and a local transmitter converts the resistance to a current. The current passes through a resistor in the panel, and the resulting voltage

Chapter 6 Meter Proving and Master Meters

In liquid flow metering systems with proving, which includes more or less all systems involved in large financial transactions, the calibration of the duty meters is established by comparing them with a reference device in the field. Uncertainty immediately after proving depends on the uncertainty of the reference and the additional uncertainties arising from making the comparison. The meter uncertainty is the dominant term in the uncertainty analysis of most liquid systems, so proving is extremely important.

The usual concept of meter proving is to calibrate each meter against a volumetric standard in the field. The volume of the proving device will vary with temperature and pressure, and the temperature and pressure at the prover and the meter will generally differ slightly. Both potentially lead to measurement bias, and should therefore be corrected even if their effects can be shown to be small.

Meter proving, whether it uses a master meter or a meter prover as the reference device, involves a limited number of repeat measurements and Type A analysis therefore applies to the results. Bearing in mind the difficulties of estimating the standard deviation from a few repeats (see 4.7) some care is needed.

Reasonable changes in process conditions must be allowed between proves. That leads to an extra uncertainty term, and a need to establish what "reasonable" in the previous sentence means.

6.1 Repeatability

For decades proving results were accepted if 5 runs repeated within a 0.05% range and it largely remains so today. Why not calculate the standard deviation? For one thing, the range idea was established before even hand calculators existed and working out a standard deviation was laborious. Mainly, though, given a very small number of repeats the standard deviation of the population can be estimated at least as accurately from the range as by direct calculation. It is strange, the author has tried in vain to find the theory behind it, but a Monte Carlo analysis shows that it is so.

MPMS 13.2 (1994) worked back from n = 5 runs with an overall meter factor range of 0.0005 to a Type A uncertainty of ±0.027%. From there, they calculated ranges for other values of n to give the same uncertainty. An alternative set of values to give an uncertainty of ±0.073% (3 runs within 0.0005) was added in MPMS 4.8 (1995). Some values are shown below.

n	0.027%	0.073%
3	0.0002	0.0005
4	0.0003	0.0009
5	0.0005	0.0014
6	0.0006	0.0017
7	0.0008	0.0021

n	0.027%	0.073%
8	0.0009	0.0025
9	0.001	0.0028
10	0.0012	0.0032
11	0.0013	0.0034
12	0.0014	0.0037

The DECC requirement for 5 runs with a maximum deviation from the mean of 0.05% is not the same, so the API uncertainty figures do not apply. Deviation from the mean is awkward, because for a given

Chapter 7 Assembling the Calculation

This chapter attempts to provide an overview of the complete procedure for carrying out an actual uncertainty evaluation, step by step. Ideas from previous chapters are brought in where necessary and the practicalities of performing each step are discussed in detail.

More mathematical concepts to extend this type of uncertainty are covered in Chapter 8. As this chapter is optional, it is at the end.

7.1 The Output Variable

The output variable may seem a strange place to begin, but until it is determined neither the input variables nor the relationship between the inputs and the output can be established. Up to now, we have assumed that it is obvious what the output variable of interest will be. However, flow computers generally provide many output variables, for example mass flow rate, batch standard volume total, and daily energy total. Deciding which one to analyse is not always as straightforward as it should be.

The simplest way to determine the relevant output variable is to check the specification. If it permits a maximum uncertainty in the batch standard volume total referenced to 60ºF of ±0.25% with a coverage factor of k=2, then that is clear. But what if part of this information is missing? All too often, it is.

As the type of flow metering system we are discussing exists mainly to count money, the output that matters most is self-evidently the one that forms the basis of the bill. In other words, flow uncertainty is ultimately measured in dollars. Bills are always based on totals: batch totals for discrete transfers such as tanker loading, and period totals for continuous transfers such as pipelines.

On this basis, it would be logical to reject flow rates as interesting output variables. This is convenient, as the frequency inputs of many flow computers have rather poor resolution. Flow computer manufacturers care about inputs; they just save their best efforts for the inputs that they know matter – in this case, the pulse count rather than the frequency.

Using totals also eliminates short-term repeatability effects. These are hard to quantify, as repeatability is only observable when one instrument is calibrated against another. They do not affect the bill as they will even out over any reasonably long interval.

If no output coverage factor is stated, then it should normally be assumed that k=2. This can be justified, for example, by quoting ISO 5168 (10.1):

It is recommended that for most applications a coverage factor, k = 2, be utilized to provide a confidence level of approximately 95 %.

If the specification does not even indicate which form of flow is of interest, the billing value can usually be deduced from the nature of the transaction, for example:

- For gas sold as fuel and shipped by pipeline, the daily energy total
- For crude oil loaded onto tankers, the batch standard volume total in barrels at 60ºF, including water correction

Chapter 8 Mathematical Extensions

This chapter is optional. It covers several topics, which are linked by their requirement for some more advanced mathematics. Both the approach and the notation used in the section are believed to be original.

8.1 Analytical Determination Sensitivity Coefficients

8.1.1 Symbols

The usual terminology of The GUM for sensitivity coefficients is simply c_i and ISO 5168 adds the option of inserting an asterisk to make a relative sensitivity coefficient c_i^*. On the other hand, the expanded uncertainty of variable x_i is represented by $U(x_i)$. There is sure to be logic behind this lack of symmetry, and so far it has caused no problems.

This section covers combining sensitivity coefficients from multiple sources and sometimes from multiple equations, which is difficult to represent clearly using the standard GUM terminology. Where it is not clear from the context, the sensitivity of y to x_i is therefore represented by $c(y/x_i)$ in this section. This notation has been chosen to be instantly distinguishable from The GUM standard, and is also quite intuitive when combined with the GUM format for uncertainty $U(x_i)$:

$$U(y) = \sqrt{[c(y/x_1) \times u(x_1)]^2 + \ldots}$$

Here you can imagine the two uses of x_1 “cancelling” and leading to an uncertainty in y. Also, where absolute (and therefore dimensional) sensitivity coefficients are used, the units of the coefficient are the units of the value within the brackets.

8.1.2 Sensitivity Rules

The rules of calculus can be used to develop some formulae that make it relatively simple to evaluate sensitivity coefficients for functions with common influences. The results are summarised in the cookbook below.

Product of two functions

If two functions f and g are multiplied together:

$$y = f(x_1, x_2, \ldots, x_n) \times g(x_1, x_2, \ldots, x_n)$$

The overall sensitivity to x_i can be calculated, and expanded using the standard formula for the differential of the product of two variables:

$$c(y/x_i) = \frac{\partial y}{\partial x_i} = f\frac{\partial g}{\partial x_i} + g\frac{\partial f}{\partial x_i} = f \cdot c(g/x_i) + g \cdot c(f/x_i)$$

The relative sensitivity is therefore:

$$c^*(y/x_i) = \frac{x_i}{y}\left(f\frac{\partial g}{\partial x_i} + g\frac{\partial f}{\partial x_i}\right)$$

Replacing y with f x g and rearranging:

$$c^*(y/x_i) = \frac{x_i}{g}\frac{\partial g}{\partial x_i} + \frac{x_i}{f}\frac{\partial f}{\partial x_i}$$

The relative sensitivity to f alone (and similarly for g) is:

$$c^*(f/x_i) = \frac{x_i}{f}\frac{\partial f}{\partial x_i}$$

The previous equation can be rearranged using this definition,. The relative sensitivity to x_i is the sum of the relative sensitivities of each of the functions f and g:

$$c_i^* = c^*(f/x_i) + c^*(g/x_i)$$

Sum of two functions

$$y = f(x_1, x_2, ..., x_n) + g(x_1, x_2, ..., x_n)$$

The absolute sensitivity to x_i is the sum of the relative sensitivities of each of the functions f and g:

$$c_i = \frac{\partial y}{\partial x_i} = \frac{\partial f}{\partial x_i} + \frac{\partial g}{\partial x_i} = c(f/x_i) + c(g/x_i)$$

The relative sensitivity to x_i is

$$c_i^* = \frac{x_i}{y}\left(\frac{\partial f}{\partial x_i} + \frac{\partial g}{\partial x_i}\right)$$

Inserting the definitions of individual relative sensitivity and replacing y with (f+g):

$$c_i^* = \frac{f \cdot c^*(f/x_i) + g \cdot c^*(g/x_i)}{f + g}$$

The relative sensitivity of the sum is therefore the weighted average of the relative sensitivity of the two components.

Function of a function

Suppose that:

$$y = f(t_1, t_2, ..., t_n)$$

And one of the terms is calculated from other variables:

$$t_i = g(x_1, x_2, ..., x_n)$$

Using the function of a function expression from A-level calculus:

$$\frac{\partial y}{\partial x_i} = \frac{\partial f}{\partial t_i} \times \frac{\partial t_i}{\partial x_i}$$

The relative sensitivity to x_i can then be written:

$$c_i^* = \frac{x_i}{y} \times \frac{\partial f}{\partial t_i} \times \frac{\partial g}{\partial x_i}$$

Multiplying by t_i / t_i and rearranging:

$$c_i^* = \frac{x_i}{y} \times \frac{\partial f}{\partial t_i} \times \frac{\partial t_i}{\partial x_i} \times \frac{t_i}{t_i} = \left(\frac{t_i}{y} \times \frac{\partial f}{\partial t_i}\right) \times \left(\frac{\partial g}{\partial x_i} \times \frac{x_i}{t_i}\right)$$

So the relative sensitivity of the function of a function is the product of the sensitivities of its two constituents:

$$c_i^* = c^*(f / t_i) \times c^*(t_i / x_i)$$

Here again, the $c(f/t_i)$ notation is intuitive: you can imagine the two t_i terms cancelling. This method is particularly handy when evaluating complicated functions, as it allows the evaluation to be performed step by step.

8.1.3 Sensitivity Cookbook

This section gives the relative sensitivity c* and absolute sensitivity c of various common functions where y is a function of one variable x. All other variables are constant, or at least not currently of interest. Results 2 – 4 are related, and could be derived from each other using function of a function, or simply by replacing variables. They are given separately for convenience.

	y =	c*(y)	c (y)
1	ax^n	n	$n\frac{y}{x}$
2	$(1+ax)^n$	$\frac{nax}{1+ax}$	$\frac{nay}{1+ax}$
3	$ax^n + b$	$\frac{nax^n}{y}$	nax^{n-1}
4	$(ax+b)^n$	$\frac{nax}{ax+b}$	$\frac{nay}{ax+b}$
5	e^x	x	y

Here are the combination rules:

	y =	c*(y/x)	c (y/x)
A	$f(x)+g(x)$	$\frac{f \cdot c^*(f/x) + g \cdot c^*(g/x)}{f+g}$	$c(f/x)+c(g/x)$
B	$f(x) \times g(x)$	$c^*(f/x)+c^*(g/x)$	$f \cdot c(g/x_i)+g \cdot c(f/x_i)$